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Laplace Transform Approach for Solving Integral Equations Using Computer Algebra System

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Abstract. The Laplace transform method, along with Computer Algebra Systems (CAS) "Maple" v. 13**,** are extremely successfully applied for solving a class of integral equations with an arbitrary order, including fractional order integral equations. The combining of both powerful approaches allows students more quickly, enjoyable and thoroughly to master the material.

1. INTRODUCTION

The Laplace transform method is a powerful approach for solving a variety of differential, integral and differ-integral equations and systems of equations. In some of the solutions *Computer Algebra System (CAS) – "Maple" v. 13* can also be used. The detailed presented solutions describe the matter of the study, and the use of the computer technologies helps the students for easier learning and for deeper understanding of the mathematical facts.

The following section is rather preliminary. It includes definitions and basic properties of the functions which we use in the present work, i.e. the Heaviside function, the Gamma functions, the Mittag-Leffler functions, as well several graphs of some of them are given.

In the third section the definitions and the basic properties of the Laplace original function, the Laplace transform and the inverse Laplace transform are given, supported by numerous examples and problems. As an illustration it is included a table of the transforms of some often used functions and they are derived the transforms of the function $f(t) = t^{\alpha}$ for complex α with $\mathrm{Re}\,\alpha > -1$, the exponential function and the Mittag-Leffler function.

The acquired knowledge is used further for finding the Laplace transform of a given original function, as well as for restoring the original function from its transform.

The last section is dedicated to various applications of operational calculus – solving of some classes of integral equations including those with fractional Riemann-Liouville integral.

2. MITTAG-LEFFLER FUNCTIONS

The *Special Functions* (SF) is a very old branch of mathematics and the search for a complete unified theory has been going since the 19th century. Its significance as an instrument of mathematical analysis is well known by the applied scientists and the engineers who deal with practical applications of differential and integral equations and systems. The variety of problems leading to an application of SF has stimulated the development of their theory. SF arise most often as solutions of some basic ODE as well as when solving PDE by the separation of variables method.

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Their definitions and well known properties can be found in the good old reference books on SF from the so called "classical era", like those of Bateman and Erdélyi (3 volumes, 1953-55); Abramowitz and Stegun (1964), Slater (1966), Luke (1969), and so on.

Let us mention that the so called *Classical SF (SF of Mathematical Physics, Functions with names)*, like the Bessel, Macdonald, Lommel, Struve functions and the other cylinder functions; the Gaussian, Kummer, Tricomi functions; all the classical orthogonal polynomials (the Laguerre, Hermite, Jacobi, Gegenbauer, Chebyshev polynomials and others); the incomplete Beta and Gamma functions; the error function; the Airy, Whittaker functions and so on, all they are connected with differential equations of *integer* order, in contrast to functions, which solve differential equations of *NON-integer (fractional)* order and are known as *Special Functions of Fractional Calculus* (***SF of FC***). A simple example of such a function is the Mittag-Leffler function, deservedly regarded as a *"Queen"* of FC (more information about this function can be seen e.g. in the books [1] – [7]). It solves a fractional order differential equations, for example the equation $D^{\alpha} y_{\alpha}(z) = \lambda y_{\alpha}(z)$, $\alpha > 0$, whose solution

$$y_{\alpha}(z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^{\alpha}) = z^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k z^{\alpha k}}{\Gamma(\alpha k + \alpha)} ,$$

is called an α-exponential function and it is a generalization of the exponent $\exp(\lambda z)$, i.e. $y_1(z) = E_{1,1}(\lambda z) = \exp(\lambda z)$.

The functions

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \tag{1}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0, \tag{2}$$

known as one-parametric and two-parametric Mittag-Leffler functions respectively, are also a generalization of the exponential function $\exp(z) = e^z$. They were introduced respectively by Mittag-Leffler and Agarwal and were studied in details by Dzherbashyan. From (2) it follows immediately that

$$E_{1,1}(z) = E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z , \tag{3}$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z^2} ,$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{e^z - 1 - z}{z^2} .$$

For $\beta = 1$ the function (2) is reduced to the one-parametric Mittag-Leffler function, i.e. $E_{\alpha,1}(z) = E_{\alpha}(z)$.

Note that the series (2) converges in the whole complex plane, i.e. it presents an entire function. In particular, the series (1) and (3) are also entire functions.

The good graphical and analytical possibilities of Computer Algebra Systems (CAS) make possible to simplify as well as to illustrate the Mittag-Leffler functions graphs. Here, this is made with the help of CAS "Maple" 13. Namely, the students consider the functions $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$ for different values of α and β.

For example, beginning with the functions $E_{\alpha,1}(z) = E_{\alpha}(z)$

> $ML\alpha := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha \cdot k + 1)}$;

$$ML\alpha := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}$$

and $E_{1,\beta}(z)$, for which we first need the formula of $E_{\alpha,\beta}(z)$, written bellow

$> ML\alpha\beta := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha \cdot k + \beta)};$

$$ML\alpha\beta := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}.$$

We simplify them:

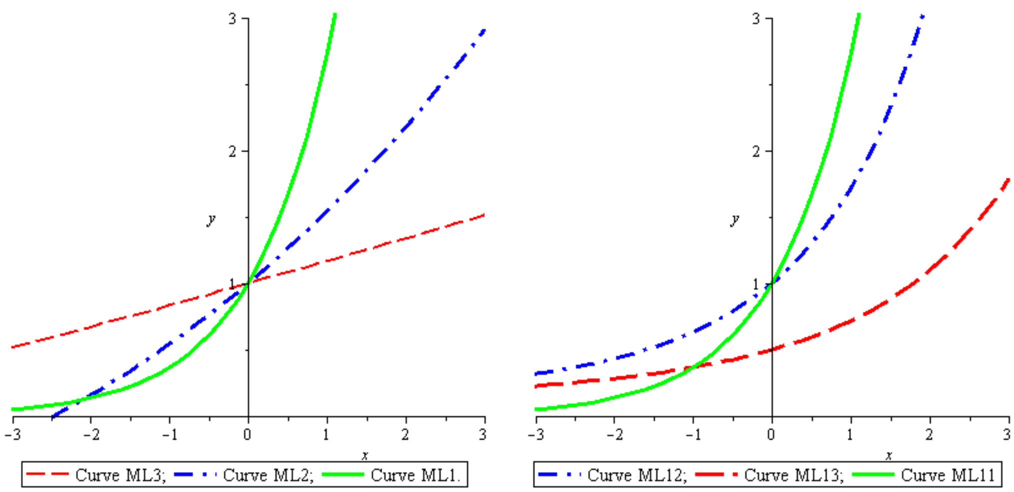
$> ML1 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}; \; ML2 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(2 \cdot k+1)}; \; ML3 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(3 \cdot k+1)};$

$$ML1 := e^x; \; ML2 := \cosh(\sqrt{x}); \; ML3 := \frac{1}{3} e^{x^{1/3}} + \frac{2}{3} e^{-\frac{1}{2} x^{1/3}} \cos\left(\frac{1}{2}\sqrt{3}\, x^{1/3}\right);$$

$> ML11 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}; \; ML12 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)}; \; ML13 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+3)};$

$$ML11 := e^x; \; ML12 := \frac{e^x - 1}{x}; \; ML13 := \frac{e^x - 1 - x}{x^2},$$

and then we plot their graphs. For example, below are shown the graphics of the functions $E_\alpha(z)$ with $\alpha = 1, 2, 3$ and $E_{1,\beta}(z)$ with $\beta = 1, 2, 3$

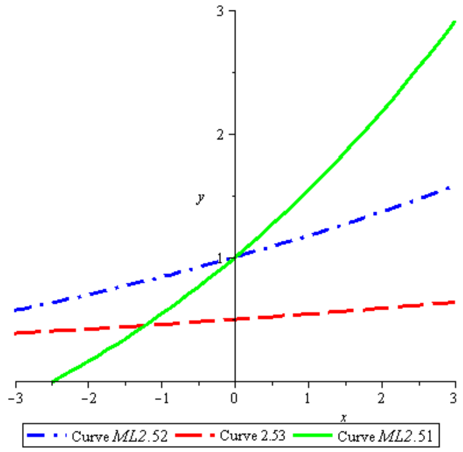


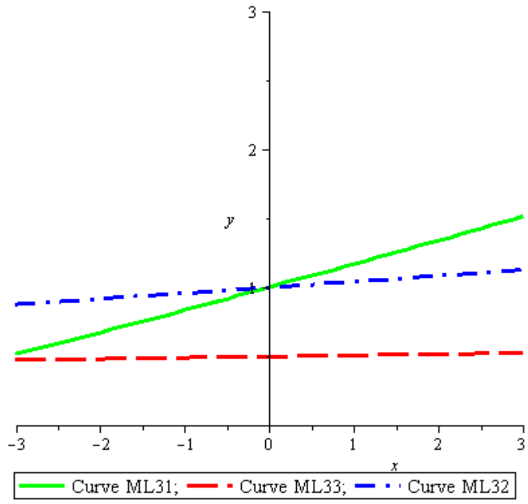
The other two graphs are for factional value of α, and β respectively with values $\beta = 1, 2, 3$, i.e. for $E_{2.5,\beta}(z)$ $(\alpha = 2.5, \; \beta = 1, 2, 3)$:

$>ML2.51 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(2.5 \cdot k+1)}; \; ML2.52 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(2.5 \cdot k+2)}; \; ML2.53 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(2.5 \cdot k+3)};$

and $E_{3,\beta}(z)$ $(\alpha = 3, \beta = 1, 2, 3)$, i.e.

$>ML31 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(3 \cdot k+1)}; \; ML32 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(3 \cdot k+2)}; \; ML33 := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(3 \cdot k+3)};$





3. LAPLACE TRANSFORM

In view of the following considerations, we give the definitions and show some of the properties of the integral Laplace transform. We note that more detailed information about this transform one can find in the books [1] – [4], [6] and [7]. Let us recall some basic facts about the Laplace transform.

The function $F(p)$ of the complex variable p, defined by the integral

$$F(p) = L(f;p) = L[f(t)](p) = \int_0^\infty e^{-pt} f(t)dt, \tag{4}$$

is said to be the Laplace transform of the function $f(t)$ which is called the original. For the existence of the integral (4) the function $f(t)$ has to be of exponential order α, which means that there exist positive constants $M > 0$ and $t > T$, such that

$$|f(t)| \le Me^{\alpha t} \quad \text{for all } t > T.$$

In other words, the function $f(t)$ must not grow faster than a certain exponential function when $t \to \infty$. Additionally, in order to be original, $f(t)$ should be zero for the negative values of the variable t and should also be in parts continuous (with its derivatives).

Remark 1. Let us mention that the integral (4) may exist even if the function $f(t)$ is not original. For example such is the function $f(t) = \frac{1}{\sqrt{t}}$ (and in general the function $f(t) = t^{-\alpha}$ for $0 < \alpha < 1$). In the applications of operational calculus there are functions for which $\lim_{t \to a} |f(t)| = \infty$ $(a \ge 0)$. These functions obviously are not originals, but for some of them their Laplace integral exists. Usually such functions are called *"singular" originals* or *pseudo-originals*, and their corresponding transforms – *"singular" transforms* or *pseudo-transforms*.

In what follows we are going to denote the Laplace transforms by uppercase letters and the originals by lowercase letters. The original $f(t)$ can be restored from the Laplace transform $F(p)$ with the help of the inverse Laplace transform

$$f(t) = L^{-1}[F(p);t] = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{pt} F(p)dp \ , \ \sigma = \operatorname{Re} p > \sigma_0, \tag{5}$$

where σ_0 lies in the right half-plane of the absolute convergence of the Laplace integral (4). In fact, the direct evaluation of the inverse Laplace transform by the formula (5) is often complicated. However, sometimes it gives useful information on the behavior of the unknown original $f(t)$ which we look for. Another important property is the uniqueness of the original which is well-known fact and it is given in the theorem below.

Theorem 1. (Uniqueness of the original) *If* $F(p)$ *is a transform of the two originals* $f_1(t)$ *and* $f_2(t)$, *then these originals coincide (i.e.* $f_1(t) = f_2(t)$ *) wherever they are continuous.*

Other useful formula referring to the Laplace transform of the convolution

$$\varphi(t) = (f * g)(t) = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau \tag{6}$$

of two functions $f(t)$ and $g(t)$, shows that it is equal to the product of their Laplace transforms, i.e.:

$$L[\varphi(t); p] = L[(f * g)(t); p] = F(p)G(p), \tag{7}$$

under the assumption that both functions $F(p) = L[f(t); p]$ and $G(p) = L[g(t); p]$ exist.

Further, recall that the integral, defined by the relation

$$_aR_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)d\tau \ \text{ with } \ \alpha > 0, \tag{8}$$

is sad to be *Riemann – Liouville fractional integral of order* $\alpha > 0$.

Remark 2. In fact, if $a = 0$, $\alpha > 0$ and $f \in C$, then the fractional integral (8) is a convolution of the kind

$$_0R_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau = \frac{x^{\alpha-1}}{\Gamma(\alpha)} * f(x). \tag{9}$$

The last equality, along with (6) and (7), turns to be extremely useful for finding the Laplace transform of the Riemann-Liouville integral. We use the property (7) for the evaluation of the Laplace transform of the Riemann-Liouville fractional integral.

Using Laplace transform, we need to know the transformations of at least some basic functions. For a start, let us adduce the table of the most often used transforms:

$$L[1; p] = \frac{1}{p}; \ L[t^n; p] = \frac{n!}{p^{n+1}}, n = 1, 2, 3, \ldots; \ L[e^{\alpha t}; p] = \frac{1}{p-\alpha};$$

$$L[\cos \alpha t; p] = \frac{p}{p^2 + \alpha^2}; \ L[\sin \alpha t; p] = \frac{\alpha}{p^2 + \alpha^2}; \tag{10}$$

$$L[\mathrm{ch} \alpha t; p] = \frac{p}{p^2 - \alpha^2}; \ L[\mathrm{sh} \alpha t; p] = \frac{\alpha}{p^2 - \alpha^2},$$

$$L[t^\alpha; p] = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}, \text{ with } \operatorname{Re} \alpha > -1,$$

where the function

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t} dt, \quad \operatorname{Re}(\alpha+1) > 0$$

is the Euler Gamma function.

To apply the Laplace transform method for solving of some special differential and integral equations (the so called fractional order equations), it is important to know the Laplace transform of $t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$, where $E_{\alpha,\beta}$ is the two-parametric Mittag-Leffler function (2). This is well known formula and we are just writing the Laplace transformation formula of the function $t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$, namely

$$L[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha});p]=\frac{p^{\alpha-\beta}}{p^{\alpha}-a} \quad \text{for} \quad \operatorname{Re} p>|a|^{\frac{1}{\alpha}} \tag{11}$$

(for more details see e.g. Podlubny [6], page 21 or [4]).

4. ARBITRARY ORDER INTEGRAL EQUATIONS

Until now we were dealing with the problem of finding the Laplace transform of an original function, as well as the inverse problem of restoring the original function from a given transform. Next, we are going to apply this knowledge to solve some classes of integral equations following the next algorithm:

1) Finding the Laplace transform of the given equation or system;
2) Finding the solution of the transformed equation or system, i.e. the transformed unknown functions;
3) Finding the originals of the solutions obtained in 2).

To find the desired transforms we will use the formulae (10) – (11), listed above.

Below are given a few integral equations which solving illustrates the deep capabilities of the Laplace transform method and CAS “Maple”, along with it. Their combining allows students more quickly, enjoyable and thoroughly to master the material.

So, we are going to find the solutions of the integral equations listed below (all the given functions Laplace originals).

a) $x(t)+a\int_0^t x(\tau)d\tau=1$

Solution. We find the Laplace transform of the given integral equation using the table of transforms and the Borel convolution theorem. Denoting $L[x(t),p]=X(p)$, we obtain the following transformed equation

$$X(p)+\frac{a}{p}X(p)=\frac{1}{p},$$

and using of CAS gives

➢ $X := solve\left(X+\frac{a}{p}\cdot X=\frac{1}{p},X\right);$

$$X:=\frac{1}{p+a}.$$

Therefore the solution of the given integral equation is the function $x(t)=e^{-at}$.

b) $\int_0^t (t-\tau)^2x(\tau)d\tau=\operatorname{sh}t-\sin t$

Solution. Denoting $L[x(t),p]=X(p)$, we find the Laplace transform of the given integral equation in the same way. The solution of the transformed equation is

$$X(p)=\frac{1}{2}\cdot\frac{p}{p^2+1}+\frac{1}{2}\cdot\frac{p}{p^2-1},$$

whence it follows that the solution of the solving problem is the function

$$x(t)=\frac{1}{2}\left(\cos t+\operatorname{ch}t\right)\cdot$$

c) $x(t)+\int\limits_0^t \sin(t-\tau)\,x(\tau)\,d\tau = 1-\cos t$

Solution. Applying the Laplace transform to the given equation and denoting $L[x(t),p]=X(p)$, we obtain the following algebraic equation

$$X(p)+\frac{1}{p^2+1}X(p)=\frac{1}{p}-\frac{p}{p^2+1},$$

from which the function $X(p)$ can be obtained. Here this is made using "Maple" and it gives the following result:

$> X := solve\left(X+\frac{1}{p^2+1}\cdot X=\frac{1}{p}-\frac{\mathrm{p}}{p^2+1}, X\right);$

$$X := \frac{1}{p\left(p^2+2\right)}.$$

The function $X(p)$ can be decomposed into a sum of partial fractions, for example by the method of undetermined coefficients, but here the result is obtained with the help of CAS – "Maple" v.13.

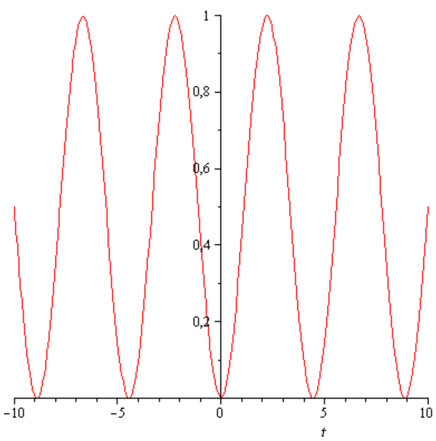
$> X := convert(X, parfrac);$

$$X := -\frac{1}{2}\frac{p}{p^2+2}+\frac{1}{2p}.$$

Now, applying the inverse Laplace transform, the solution of the given integral equation is $x(t)=\frac{1}{2}\left(1-\cos\sqrt{2}\,t\right)$.

The solution is plotted below.

$> plot\left(\frac{1}{2}\cdot(1-\cos(\mathrm{sqrt}(2)\cdot t)), t=-10..10\right);$



d) $x(t)+a\;{}_0R_t^{\alpha}x(t)=f(t),\ \alpha>0,$

where a is a real or complex parameter, ${}_0R_t^{\alpha}x(t)$ is the Riemann-Liouville fractional integral of order $\alpha>0$ defined by (8) and the function $f(t)$ is a given Laplace original.

Solution. The given equation is the well known Abel integral equation of the second kind. As a matter of fact, it can also be written in the following alternative way

$$x(t)+\frac{a}{\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1}\,x(\tau)d\tau = f(t),\ \alpha>0.$$

By applying the Laplace transform to both sides of the equation and according to Borel convolution theorem, we find

$$L[x(t);p]+\frac{a}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{p^{\alpha}}L[x(t);p]=L[f(t);p],$$

from where it follows that

$$L[x(t),p]=\frac{p^{\alpha}}{p^{\alpha}+a}L[f(t);p]=p\frac{p^{\alpha-1}}{p^{\alpha}+a}L[f(t);p].$$

Now, applying the inverse Laplace transform, and again according to the Borel theorem, formula (11) and the rule of differentiation of original function, we obtain that the solution of the integral equation has the form

$$x(t)=\frac{d}{dt}\int_0^t E_{\alpha,1}(-a\tau^{\alpha})f(t-\tau)d\tau.$$

e) $x(t)+\frac{a}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)d\tau=\cos(t)$,

where $\alpha>0$, a is a real or complex parameter.

Solution. The given equation is also the well known Abel integral equation of the second kind. Again, by applying the Laplace transform to both sides of the equation and according to Borel convolution theorem, we obtain

$$L[x(t),p]+\frac{a}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{p^{\alpha}}L[x(t);p]=\frac{p}{p^2+1},$$

which gives

$$L[x(t),p]=\frac{p^{\alpha}}{p^{\alpha}+a}\cdot\frac{p}{p^2+1}=p\cdot\frac{p^{\alpha-1}}{p^{\alpha}+a}\cdot\frac{p}{p^2+1}\cdot$$

Now, by applying the inverse Laplace transform, the Borel theorem, formula (11) and the rule of differentiation of original function, we conclude that the solution of the given equation has the form

$$x(t)=\frac{d}{dt}\int_0^t E_{\alpha,1}(-a\tau^{\alpha})\cos(t-\tau)d\tau\cdot$$

f) $x(t)+a\ {}_0R_t^{\alpha}x(t)=1+t$, $\alpha>0$,

Solution. Denoting $L[x(t),p]=X(p)$ we have

$$\frac{p^{\alpha}+a}{p^{\alpha}}X=\frac{1}{p}+\frac{1}{p^2}\cdot \tag{12}$$

Then, on the one hand (12) implies

$$X=\frac{p^{\alpha-1}}{p^{\alpha}+a}+\frac{p^{\alpha-2}}{p^{\alpha}+a},$$

and then, from the formula (11), the solution of the given equation is

$$x(t)=E_{\alpha,1}(-at^{\alpha})+t\,E_{\alpha,2}(-at^{\alpha})=E_{\alpha}(-at^{\alpha})+t\,E_{\alpha,2}(-at^{\alpha})\cdot \tag{13}$$

On the other hand (12) implies

$$X=p\cdot\frac{p^{\alpha-1}}{p^{\alpha}+a}\left(\frac{1}{p}+\frac{1}{p^2}\right),$$

and then the searched solution can be written as follows:

$$x(t)=\frac{d}{dt}\left(E_{\alpha}(-at^{\alpha})*(1+t)\right)=\frac{d}{dt}\int_0^t E_{\alpha}(-a\tau^{\alpha})(1+t-\tau)d\tau=\int_0^t E_{\alpha}(-a\tau^{\alpha})d\tau+E_{\alpha}(-at^{\alpha}).$$

Since $\int_0^t \tau^{\alpha k} d\tau = \frac{t^{\alpha k+1}}{\alpha k+1}$, for $k = 0,1,2,...$, the above integral can be evaluated, directly integrating the Mittag-Leffler function $E_\alpha(-at^\alpha)$. The result from integration is written below

$$\int_0^t E_\alpha(-a\tau^\alpha)d\tau = \sum_{k=0}^{\infty} \frac{(-a)^k t^{\alpha k+1}}{(\alpha k+1)\Gamma(\alpha k+1)} = t\sum_{k=0}^{\infty} \frac{(-at^\alpha)^k}{\Gamma(\alpha k+2)} = tE_{\alpha,2}(-at^\alpha).$$

Therefore the solution $x(t)$ can be expressed as follows

$$x(t) = tE_{\alpha,2}(-at^\alpha) + E_\alpha(-a\tau^\alpha),$$

which is the same as (13).

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